MIM JOURNAL 2018 illuminate your work
Made in Millersville Journal | www.mimjournal.com

An A-Infinity Coalgebra Structure On The Cellular Chains Of A Polygon

Quinn Minnich

Abstract

In my project I studied A-infinity coalgebra structures, which are a topic in algebraic topology in mathematics. An A-infinity coalgebra is defined by a differential graded vector space, which in this case was an n-sided polygon, and a set of coassociators Δ2, Δ3,…$ that map elements of the vector space to tensor products. In order to meet the definition of an A-infinity coalgebra, each Δn must satisfy a relation derived from a chain map from associahedra. Since each relation involves all previous Δn's, and since the relation must hold for all elements in the vector space, verifying that a particular set of coassociators meets the conditions of an A-infinity coalgebra can be difficult. Examples of such structures have been found before, but they have all required an infinite number of non-trivial Δn's. In this project, we present the first known example of an A-infinity coalgebra that uses only a finite number of non-trivial Δn's, and for which all other Δn's vanish.

Consider a polygon with n sides. Then each of the vertices, edges, and the single face of the polygon can be thought of as vectors with a dimension of zero, one, or two respectively, which we call the cellular chains of the polygon. We then define a boundary operator, ∂ which when applied to a vector gives its boundary (for example, applying ∂ to one of the edges would yield the two vertices that make up the ends of the edge). This boundary operator, along with the cellular chains of the polygon, define a differential graded vector space.

Now once we have a differential graded vector space, we can proceed to define an operation which we will denote as Δ2. Δ2 can be thought of as a means of "factoring" a vector—specifically, applying Δ2 to any vector yields a tensor product of two vectors as a result. In a similar way, Δ3 is defined to return a tensor product of three vectors, Δ4 at tensor product of four, and so on, and it is this sequence of Δn's, along with the differential graded vector space, that define an A-infinity coalgebra structure.

In order to actually form an A-infinity coalgebra structure however, each Δn must obey a certain relation that arises from a chain map between the differential graded vector space and a sequence of geometrical objects called associahedra. Basically, there is a relation for each of the

Δn's, and each relation connects its particular Δn to all the other Δn's that came before. Intuitively, the relations can be thought of in the following manner. Suppose you took a random vector from the vector space (such as one of the edges), and first applied Δn to it, followed by ∂. Then suppose you applied ∂ followed by Δn. If the results are not equal, then pick any terms that differed and their sum will form one side of the equation. To calculate the other side, we do a similar thing with the rest of the lower Δn, except that the sum of their indexes have to equal on more than n. So for example, if the first side of the equation was based off of $\Delta 6$, then the other side would be formed from the terms generated by applying $\Delta 2$ followed by $\Delta 5$, applying $\Delta 3$ followed by another $\Delta 4$, applying $\Delta 4$ followed by $\Delta 3$, and applying $\Delta 5$ followed by $\Delta 2$. The reason for this is because $\Delta 6$ produces a tensor product of six vectors, and so we must apply only sequences of lower Δn that give similar terms (for example, $\Delta 5$ produces a tensor product of five vectors, and then $\Delta 2$ splits one of those into two, resulting in six vectors). Since associahedra help to keep track of all these combinations, defining the relations in terms of a chain map to the associahedra helps to write the equations in a compact form.

The key component of these relations is that once a particular Δn is defined, more conditions and restrictions are imposed on the next Δn up. If a rule can be created that shows how to define each Δn so that all of the relations hold, then the result qualifies as an of A-infinity coalgebra structure. Currently, there are known examples of A-infinity coalgebra structures which meet these conditions, but that require an infinite number of Δn's, since each is induced by the ones that came before it. My result is the first known example in which only n Δn are needed, and the rest could safely send everything to zero without messing up any of the relations.

What I found most interesting about this project wasn't so much the final result itself, but the journey that led to it. Originally my advisor, Dr. Ron Umble, and I were discussing a different topic altogether, and as an exercise I was asked to find the Δn operations for a particular polygon with a small number of sides. I was already given $\Delta 2$, which had been defined by another student of his several years back (Kravatz, 2008), and so I only needed to find $\Delta 3$ and up. As I worked on the concrete example, I discovered a pattern in how each Δn could be defined, and speculated that the pattern could be generalized for a polygon of any number of sides. My advisor encouraged me to examine the topic further, and I set out to prove that my generalization did in fact hold for all polygons.

Unfortunately, proving a result for a generic polygon was far more difficult. With a concrete number of sides, there are a finite number of relations to verify, and the process can be done by hand. As the number of sides increased, however, the computations became exponentially more numerous, and even with the help of a computer I was unable to directly verify the pattern for polygons with more than twenty sides. An added complication was that each additional side required the construction of an additional Δn, which in turned required verifying an additional, more complicated relation, and so in order to prove my result, I needed to somehow show that my pattern would always satisfy the relations without assuming anything about either the number of sides, or the number of relations being verified.

Since this problem involved two unknowns (the number of sides and the number of relations), my advisor suggested approaching the problem through induction.

One option was to prove the first relation by hand for a generic number of sides, and then show that each of the other relations would follow from the last. Alternately, I could also prove all of the relations for a polygon of only a few sides, and then use induction to show that that a polygon with one more side would also satisfy the relations. Neither of these approaches were easy for me to follow, however, and so in the end I used an entirely different type of proof that didn't rely on induction at all.

In my approach, I first noted that the way I had defined my Δn's allowed each relation to be greatly simplified so that it depended only on the current Δn, the previous one, Δ2, and the boundary operator. Since each Δn when applied to a vector produced a tensor product of vectors, verifying the relations boiled down to somehow showing that all terms generated by these Δn on both sides of the equation canceled out. To do this, I studied the types of terms generated by each Δn, and discovered that they could each be divided into one of five classes based on certain characteristics of the tensor product. I then showed that the five classes canceled each other out, and thereby proved the relations without assuming anything about number of sides in the polygon, or the particular relation I had verified. This meant that the proof applied to all relations and polygons, which gave me my result.

While the proof itself was a fun challenge that took me a while to properly iron out, I was still far from done. Throughout all my research, I had been making a simplifying assumption that I was working in mod two, which essentially meant that I could ignore signs. In order to make my result interesting to the rest of the mathematical world, however, I had to generalize my result to any field and show that I still had an A-infinity coalgebra structure even when I accounted for signs. While this required no major changes to the structure of my proof, it was still a rather tedious process to insert all the signs and make sure that the steps all worked out. I also proved a corollary in which I removed an assumption about the polygon that helped to fix how Δ2 was defined, and my advisor and I even discovered an application in which we were able to define an A-infinity coalgebra structure on Klein bottles and certain other topological surfaces that could be decomposed into polygons.

Despite the progress we were able to make in this field, there is still much to do that could expand on this project. For one, the particular A-infinity coalgebra structure I defined was fairly easy to work with because each of the relations simplified to depend only on a few of the Δn, and so it would be interesting to see another example where each relation made use of all the Δn before it. Additionally, I used a very simple differential graded vector space with only one two-dimensional face and no vectors of any higher dimension. It would be particularly interesting to see an example of an A-infinity coalgebra structure defined using a three-dimensional solid or higher.

In conclusion, my advisor and I were rewarded with seeing the first example of an A-infinity coalgebra structure with a finite number of non-vanishing Δn's. This result has still undiscovered applications in topology, utilized an unconventional proof technique, and serves to further illustrate the beautiful patterns and connections that arise in the world of mathematics. While the result itself is only defined for a very simple type of graded vector space, it could very well serve as the first step in a long chain of further discoveries and generalizations in this area, which will no doubt continue to impact and provide meaningful applications to its field.

References

Kravatz, D. (2008). *Diagonal Approximations on an n-gon and the cohomology Ring of Closed Compact Orientable Surfaces* (Senior Thesis). Millersville University Department of Mathematics

Recommended Citation
Minnich, Q. (2018). An a-infinity coalgebra structure on the cellular chains of a polygon. *Made in Millersville Journal*, 2018. Retrieved from https://www.mimjournal.com.